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Zero Entropy Systems

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Abstract This paper introduces the notion of entropy dimension to measure the complexity of zero entropy dynamical systems, including the probabilistic and the topological versions. These notions are isomorphism invariants for measure-preserving transformation and continuity. We discuss basic propositions for entropy dimension and construct some examples to show that the topological entropy dimension attains any value between 0 and 1. This paper also gives a symbolic subspace to achieve zero topological entropy, but with full entropy dimension.

Keywords Entropy dimension · Conditional entropy · Dynamical systems · Power rule · Affinity · Symbolic dynamics

1 Introduction

In 1958, Kolmogorov applied the notion of entropy from information theory to ergodic theory. Since then, the notion of entropy has played an important role in the study of dynamical systems. Entropies measure the complexity of a dynamical system, and have been extensively studied for different maps (see also [1, 3, 9, 12, 16, 21, 23, 29]). The two main types of entropy are measure-theoretic (or metric) entropy and topological entropy. The former measures the maximal loss of information of the iteration of finite partitions in a measure-preserving transformation. The latter measures the maximal exponential growth rate of orbits for an arbitrary topological dynamical system. These two notions are connected by a so-called variational principle, which states that the topological entropy is the supremum of the metric entropies for all invariant probability measures of a given topological system (see

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[13, 14, 26]). Various authors have introduced several refinements to the notion of entropy, including slow entropy [17], measure-theoretic complexity [10], and entropy convergence rates [2].

Moreover, the local structure of maps can be deeply characterized by defining entropy pairs, entropy tuples, entropy sets, or entropy points in both topological and measure-theoretical situations. A series of works has been published to study the connection between measure-theoretic entropy notions and topological entropy ones. These studies investigate their local behavior, and in many cases, establish new variational principles. See [9, 14] and [26] for related propositions and results.

Although systems with positive entropy are much more complicated than those with zero entropy, zero entropy systems have various complexity, and have been studied by [5, 7, 8, 11, 13, 15, 18, 19, 22]. These authors adopted various methods to classify zero dynamical systems. Carvalho [5] introduced the notion of entropy dimension to distinguish the zero topological entropy systems and obtained some basic properties of entropy dimension. Ferenczi and Park [11] investigated a new entropy-like invariant for the action of $\mathbb{Z}$ or $\mathbb{Z}^d$ on a probability space.

This paper will consider the zero entropy systems, including probabilistic and topological versions. Firstly, we study properties of the conditional entropy dimension for an invariant partition. Results show that the conditional entropy dimension function satisfies an inequality between those of the transformation T and its power T^r (r is any positive integer). This paper also proves an affinity property for the invariant measures. Secondly, we introduce the generalized s-topological entropy and topological entropy dimension of a topological dynamical system. We give the formulas of calculating the s-topological entropy of the piecewise monotone continuous map on the unit interval and the shift map on the subshift of symbolic spaces. The analysis in this study also obtains the relationship between the s-topological entropies of the topological semi-conjugate systems. This paper calculates the entropy dimensions of some examples, such as the homeomorphism of the unit circle and the tent maps. We also construct some examples to show that every number in (0, 1) can be attained by the entropy dimensions of the dynamical systems and a dynamical system whose entropy dimension is one and topological entropy is zero, which answers a question asked by Carvalho [5]. In the end of those works, we mention that the well-known variational principle is not true for entropy dimension from Dou, Huang, and Park's recent work [8].

2 Probabilistic Version of Entropy Dimension

In dynamical systems and ergodic theory, it is understood that a reasonable measure-theoretic or topological entropy should be a measure of the uncertainty of the system. These entropies should be invariant under measurable or topological change of coordinates, respectively. This section reviews the concept of conditional entropy on a probability space and then introduces the conditional entropy dimension.

2.1 Basic Definitions

Let $(X, \mathcal{B}, \mu)$ be a probability space. A finite partition α of this probability space is a disjoint collection $\alpha = \{A_1, A_2, \dots, A_k\}$ of measurable sets whose union is all of X. The entropy of α is the value $H_\mu(\alpha) = -\sum_{i=1}^{k} \mu(A_i) \log \mu(A_i)$.

Given partitions $\alpha = \{A_1, A_2, \dots, A_k\}$ and $\beta = \{B_1, B_2, \dots, B_l\}$ of X, let

$$\alpha \vee \beta = \{A_i \cap B_j \mid 1 \le i \le k, 1 \le j \le l\}$$

denote their join between α and β.

We also define the conditional entropy of α given β to be the number

$$H_\mu(\alpha \mid \beta) = -\sum_{j=1}^{l} \mu(B_j) \sum_{i=1}^{k} \frac{\mu(A_i \cap B_j)}{\mu(B_j)} \log \frac{\mu(A_i \cap B_j)}{\mu(B_j)}$$

omitting the j-terms when $\mu(B_j) = 0$.

Definition 2.1 Suppose $(X, \mathcal{B}, \mu)$ is a probability space.

(a) A transformation $T : X \to X$ is measurable if $T^{-1}(\mathcal{B}) \subset \mathcal{B}$.
(b) A transformation $T : X \to X$ is measure-preserving if T is measurable and $\mu(T^{-1}(B)) = \mu(B), \forall B \in \mathcal{B}$. Thus, this type of measure μ is called an invariant measure for T.

Definition 2.2 If α is a finite partition of X and T is a continuous endomorphism from X to itself, then $T^{-i}\alpha$ is another partition of X consisting of all sets $T^{-i}A$ where $A \in \alpha$.

Proposition 2.1 *Let $(X, \mathcal{B}, \mu)$ be a probability space. If α, β are finite partitions of X, then*

(1) $H_\mu(\alpha \vee \beta) = H_\mu(\alpha) + H_\mu(\beta \mid \alpha)$.
(2) $H_\mu(\alpha \vee \beta) \le H_\mu(\alpha) + H_\mu(\beta)$.
(3) *If T is measure-preserving from X to X, then*

$$H_\mu(T^{-1}\alpha \mid T^{-1}\beta) = H_\mu(\alpha \mid \beta).$$

When μ is an invariant measure of T and $\mathcal{A}$ is a T-invariant finite partition of X, i.e. $T^{-1}\mathcal{A} = \mathcal{A}$, it is not hard to see that $H_\mu(\alpha^n \mid \mathcal{A})$ is a non-negative sub-additive sequence for any given finite partition α of X. Therefore, the conditional entropy of α with respect to $\mathcal{A}$ under T is the value

$$h_\mu(T, \alpha \mid \mathcal{A}) = \lim_{n\to\infty} \frac{1}{n} H_\mu(\alpha^n \mid \mathcal{A}) = \inf_{n\ge 1} \frac{1}{n} H_\mu(\alpha^n \mid \mathcal{A}),$$

where

$$\alpha^n = \bigvee_{i=0}^{n-1} T^{-i}\alpha = \{A_{i_0} \cap T^{-1}A_{i_1} \cap \cdots \cap T^{-(n-1)}A_{i_{n-1}} : A_{i_j} \in \alpha\}$$

and the conditional measure-theoretic entropy of T with respect to μ under T-invariant partition $\mathcal{A}$ is defined as

$$h_\mu(T \mid \mathcal{A}) = \sup_\alpha h_\mu(T, \alpha \mid \mathcal{A}),$$

where α is any finite partition of X.

We say (α, T) is a stochastic process on the probability space $(X, \mathcal{B}, \mu)$ and $(X, \mathcal{B}, \mu, T)$ is a dynamical system, which mean that T is a measure-preserving transformation of $(X, \mathcal{B}, \mu)$ to itself and α is a partition of X. Zero entropy essentially means that the effect of dynamics produces only a small growth of uncertainty of the partition of X. The conditional metric entropy dimension of this stochastic process (α, T) with T-invariant partition $\mathcal{A}$ can

be defined as follows:
First, let

$$D(s,\mu,T,\alpha \mid \mathcal{A}) = \limsup_{n\to\infty} \frac{1}{n^s} H_\mu\left(\bigvee_{i=0}^{n-1} T^{-i}\alpha \mid \mathcal{A}\right).$$

If this T-invariant partition $\mathcal{A}$ is the whole space X, then define

$$D(s,\mu,T,\alpha) = \limsup_{n\to\infty} \frac{1}{n^s} H_\mu\left(\bigvee_{i=0}^{n-1} T^{-i}\alpha\right).$$

The graph of $D(s,\mu,T,\alpha \mid \mathcal{A})$ against s shows that there is a critical value of s, $0 \le s \le 1$, at which $D(s,\mu,T,\alpha \mid \mathcal{A})$ jumps from ∞ to 0 and define

$$D(s,\mu,T \mid \mathcal{A}) = \sup_\alpha D(s,\mu,T,\alpha \mid \mathcal{A}) = \sup_\alpha\left\{\limsup_{n\to\infty} \frac{1}{n^s} H_\mu\left(\bigvee_{i=0}^{n-1} T^{-i}\alpha \mid \mathcal{A}\right)\right\}.$$

Thus, conditional entropy dimension is defined as the value

$$\begin{aligned} D(\mu,T \mid \mathcal{A}) &= \inf\Big\{s>0 : \sup_\alpha D(s,\mu,T,\alpha \mid \mathcal{A}) = 0\Big\} \\ &= \inf\left\{s>0 : \sup_\alpha \limsup_{n\to\infty} \frac{1}{n^s} H_\mu\left(\bigvee_{i=0}^{n-1} T^{-i}\alpha \mid \mathcal{A}\right) = 0\right\}. \end{aligned}$$

If $\mathcal{A}$ is the whole space X, then denote $D(\mu,T) = D(\mu,T \mid \mathcal{A})$ and this is called the measure-theoretic entropy dimension. See [7, 8] and [25].

2.2 Some Propositions

Obviously, this conditional entropy dimension we defined is a measure-theoretic isomorphic invariant. Thus, we can consider those two entropy zero dynamical systems as being not the same or being not equivalent by different conditional entropy dimension. The basic proposition of conditional entropy dimension is the power rule. The inequality of the power rule can be showed as follows.

Proposition 2.2 *For each positive integer r and $0 \le s \le 1$, we have*

$$D(s,\mu,T^r \mid \mathcal{A}) \le r^s \cdot D(s,\mu,T \mid \mathcal{A})$$

and

$$D(\mu,T^r \mid \mathcal{A}) \le r \cdot D(\mu,T \mid \mathcal{A}).$$

Proof First show that

$$D\left(s,\mu,T^r,\bigvee_{i=0}^{r-1} T^{-i}\alpha \mid \mathcal{A}\right) \le r^s \cdot D(s,\mu,T,\alpha \mid \mathcal{A})$$

for any finite partition α.

Since

$$D\left(s,\mu,T^r,\bigvee_{i=0}^{r-1}T^{-i}\alpha \mid \mathcal{A}\right) = \limsup_{n\to\infty}\frac{r^s}{(nr)^s}H_\mu\left(\bigvee_{j=0}^{n-1}T^{-rj}\left(\bigvee_{i=0}^{r-1}T^{-i}\alpha\right)\Big|\mathcal{A}\right)$$

$$= \limsup_{n\to\infty}\frac{r^s}{(nr)^s}H_\mu\left(\bigvee_{i=0}^{nr-1}T^{-i}\alpha \mid \mathcal{A}\right)$$

$$= r^s\cdot\limsup_{n\to\infty}\frac{1}{(nr)^s}H_\mu\left(\bigvee_{i=0}^{nr-1}T^{-i}\alpha \mid \mathcal{A}\right)$$

$$\le r^s\cdot D(s,\mu,T,\alpha \mid \mathcal{A}),$$

thus,

$$D(s,\mu,T^r,\alpha \mid \mathcal{A}) \le D\left(s,\mu,T^r,\bigvee_{i=0}^{r-1}T^{-i}\alpha \mid \mathcal{A}\right) \le r^s\cdot D(s,\mu,T,\alpha \mid \mathcal{A})$$

for any partition α. This implies that

$$D(s,\mu,T^r \mid \mathcal{A}) \le r^s\cdot D(s,\mu,T \mid \mathcal{A})$$

and we conclude that

$$D(\mu,T^r \mid \mathcal{A}) \le r\cdot D(\mu,T \mid \mathcal{A}). \qquad \square$$

This type of conditional entropy dimension shares the property of affinity, and its proof is a direct result from ergodic theory.

Theorem 2.1 (Affinity) *Let $\mathcal{A}$ be a T-invariant partition of the dynamical system $(X,\mathcal{B},\mu,T)$. Then the map $\mu\to D(s,\mu,T,P \mid \mathcal{A})$ is affine where P is any finite partition of X. Hence, so is the map $\mu\to D(s,\mu,T \mid \mathcal{A})$. I.e.*

$$D(s,\mu,T,P \mid \mathcal{A}) = \lambda\cdot D(s,\mu_1,T,P \mid \mathcal{A}) + (1-\lambda)\cdot D(s,\mu_2,T,P \mid \mathcal{A})$$

and

$$D(s,\mu,T \mid \mathcal{A}) = \lambda D(s,\mu_1,T \mid \mathcal{A}) + (1-\lambda)D(s,\mu_2,T \mid \mathcal{A}),$$

where μ, μ_1 and μ_2 are invariant probability measures with

$$\mu = \lambda\cdot\mu_1 + (1-\lambda)\cdot\mu_2$$

and $0<\lambda<1$.

Proof For any integer n, constant $0<\lambda<1$ and invariant probability measures μ,μ_1,μ_2, with

$$\mu = \lambda\mu_1 + (1-\lambda)\mu_2.$$

Using the concave property of the function $f(x) = -x\log x$ and some calculation, we obtain the following inequality for any $A\in P$,

$$
\begin{aligned}
0 &\le -\mu(A)\log\mu(A) + \lambda\mu_1(A)\log\mu_1(A) + (1-\lambda)\mu_2(A)\log\mu_2(A)\\
&= -\lambda\mu_1(A)\{\log\mu(A) - \log\lambda\mu_1(a)\} - (1-\lambda)\mu_2(A)\{\log\mu(A) - \log(1-\lambda)\mu_2(A)\}\\
&\quad - \mu_1(A)\lambda\log\lambda - \mu_2(A)(1-\lambda)\log(1-\lambda)\\
&\le -\mu_1(A)\lambda\log\lambda - \mu_2(A)(1-\lambda)\log(1-\lambda).
\end{aligned}
$$

The summation of this inequality for all $A \in P$ and the concavity of $-x\log x$ lead to

$$0 \le H_\mu(P^n) - \lambda H_{\mu_1}(P^n) - (1-\lambda)H_{\mu_2}(P^n) \le -\lambda\log\lambda - (1-\lambda)\log(1-\lambda) \le \log 2.$$

Hence,

$$D(s,\mu,T,P) = \lambda D(s,\mu_1,T,P) + (1-\lambda)D(s,\mu_2,T,P).$$

Next, consider the finite partition P and T-invariant partition $\mathcal{A}$. We have the formula

$$H_\mu(P \mid \mathcal{A}) = H_\mu(P \vee \mathcal{A}) - H_\mu(\mathcal{A}).$$

Using these finite partitions P^n and $\mathcal{A}$, we have

$$0 \le H_\mu(P^n \vee \mathcal{A}) - \lambda H_{\mu_1}(P^n \vee \mathcal{A}) - (1-\lambda)H_{\mu_2}(P^n \vee \mathcal{A}) \le \log 2 \tag{2.1}$$

and

$$0 \ge -[H_\mu(\mathcal{A}) - \lambda H_{\mu_1}(\mathcal{A}) - (1-\lambda)H_{\mu_2}(\mathcal{A})] \ge -\log 2. \tag{2.2}$$

Since the second term of (2.2) is non-positive, adding it to the second term of (2.1) does not increase the latter's value. Thus

$$H_\mu(P^n \mid \mathcal{A}) - \lambda H_{\mu_1}(P^n \mid \mathcal{A}) - (1-\lambda)H_{\mu_2}(P^n \mid \mathcal{A}) \le \log 2.$$

Similarly, adding the second term of (2.1) to that of (2.2) does not decrease the latter's value, so

$$-\log 2 \le H_\mu(P^n \mid \mathcal{A}) - \lambda H_{\mu_1}(P^n \mid \mathcal{A}) - (1-\lambda)H_{\mu_2}(P^n \mid \mathcal{A}).$$

Putting these two inequalities together gives

$$-\log 2 \le H_\mu(P^n \mid \mathcal{A}) - \lambda H_{\mu_1}(P^n \mid \mathcal{A}) - (1-\lambda)H_{\mu_2}(P^n \mid \mathcal{A}) \le \log 2.$$

Now, dividing by n^s and taking $\limsup$ when $n \to \infty$ gives that

$$D(s,\mu,T,P \mid \mathcal{A}) = \lambda D(s,\mu_1,T,P \mid \mathcal{A}) + (1-\lambda)D(s,\mu_2,T,P \mid \mathcal{A})$$

as required. Therefore, we have for all partitions P,

$$
\begin{aligned}
\sup_P D(s,\mu,T,P \mid \mathcal{A}) &= \sup_P\{\lambda D(s,\mu_1,T,P \mid \mathcal{A}) + (1-\lambda)D(s,\mu_2,T,P \mid \mathcal{A})\}\\
&\le \sup_P \lambda D(s,\mu_1,T,P \mid \mathcal{A}) + \sup_P (1-\lambda)D(s,\mu_2,T,P \mid \mathcal{A}),
\end{aligned}
$$

which implies

$$D(s,\mu,T \mid \mathcal{A}) \le \lambda D(s,\mu_1,T \mid \mathcal{A}) + (1-\lambda)D(s,\mu_2,T \mid \mathcal{A}).$$

The other side of inequality is shown next.

For all $\epsilon > 0$, we can choose partitions P_1, P_2 such that

$$D(s,\mu_1,T,P_1 \mid \mathcal{A}) > D(s,\mu_1,T \mid \mathcal{A}) - \epsilon \text{ and } D(s,\mu_2,T,P_2 \mid \mathcal{A}) > D(s,\mu_2,T \mid \mathcal{A}) - \epsilon.$$

Thus,

$$\begin{aligned} D(s,\mu,T \mid \mathcal{A}) &\geq D(s,\mu,T,P_1 \vee P_2 \mid \mathcal{A}) \\ &= \lambda D(s,\mu_1,T,P_1 \vee P_2 \mid \mathcal{A}) + (1-\lambda) D(s,\mu_2,T,P_1 \vee P_2 \mid \mathcal{A}) \\ &\geq \lambda D(s,\mu_1,T,P_1 \mid \mathcal{A}) + (1-\lambda) D(s,\mu_2,T,P_2 \mid \mathcal{A}) \\ &> \lambda (D(s,\mu_1,T \mid \mathcal{A}) - \epsilon) + (1-\lambda)(D(s,\mu_2,T \mid \mathcal{A}) - \epsilon) \\ &= \lambda D(s,\mu_1,T \mid \mathcal{A}) + (1-\lambda) D(s,\mu_2,T \mid \mathcal{A}) - \epsilon, \end{aligned}$$

which implies

$$D(s,\mu,T \mid \mathcal{A}) \geq \lambda D(s,\mu_1,T \mid \mathcal{A}) + (1-\lambda) D(s,\mu_2,T \mid \mathcal{A}). \qquad \square$$

From the definition of conditional entropy dimension and Theorem affinity, we obtain the following.

Corollary 1 *If the notions and conditions are the same as in Theorem affinity, then*

$$D(\mu,T \mid \mathcal{A}) = \max\{D(\mu_1,T \mid \mathcal{A}), D(\mu_2,T \mid \mathcal{A})\}.$$

3 Topological Version of s-topological Entropy

Topological entropy is one of the most fundamental dynamical invariants associated with a continuous map. It roughly measures the orbit structure complexity of the map. Zero entropy generally means that the continuous map presents a smaller complicated dynamical behavior somewhere in the space. The s-topological entropy dimension of a dynamical system is introduced in this section. After that, we give the formulas for calculating the s-topological entropy of the piecewise monotone continuous map and the subshift of symbolic dynamics.

3.1 Some Definitions

Let (X,d) be a compact metric space and α an open cover of X. Denote by $\aleph(\alpha)$ the number of sets in a finite subcover of α with smallest cardinality.

Definition 3.1 Let $T : X \to X$ be a continuous map and $s \geq 0$ a real number. The s-topological entropy of T is defined as

$$D(s,T) = \sup_{\beta} \limsup_{n\to\infty} \frac{1}{n^s} \log \aleph\left(\bigvee_{i=0}^{n-1} T^{-i}\beta\right), \tag{3.1}$$

where β ranges over all open covers of X and

$$\bigvee_{i=0}^{n-1} T^{-i}\beta = \{A_{i_0} \cap T^{-1}A_{i_1} \cap \cdots \cap T^{-(n-1)}A_{i_{n-1}} : A_{i_j} \in \beta\}.$$

When $s=1$, $D(s,T)$ is just the topological entropy of T (usually denoted by $h(T)$), which can be represented as

$$h(T)=\sup_{\text{open cover }\beta}\lim_{n\to\infty}\frac{1}{n}\log\aleph\left(\bigvee_{i=0}^{n-1}T^{-i}\beta\right)$$

because the limit of $\frac{1}{n}\log\aleph(\bigvee_{i=0}^{n-1}T^{-i}\beta)$ exists as $n\to\infty$ for any open cover β (see [28]).

Let $n\in\mathbb{N}$ and define a new (Bowen) metric d_n on X,

$$d_n(x,y)=\max_{0\le i<n}d(T^ix,T^iy),$$

where $x,y\in X$. This metric leads to the following definitions.

Definition 3.2 A set $F\subset X$ is called (n,ϵ)-spanning set of X for T if for any $x\in X$, there exists $y\in F$ with $d_n(x,y)\le\epsilon$. The dual definition is as follows. A set $S\subset X$ is called (n,ϵ)-separated set of X for T if $d_n(x,y)>\epsilon$ for every pair of distinct point $x,y\in S$, $x\neq y$.

Denote

$$r(n,\epsilon,X)=\min\{\#F:F\subset X\text{ is a }(n,\epsilon)\text{-spanning set for }T\}$$

and

$$s(n,\epsilon,X)=\max\{\#S:S\subset X\text{ is a }(n,\epsilon)\text{-separated set for }T\},$$

where $\#S$ is the number of elements in S.

From [5], we have the following property, which can be immediately obtained from the inequalities between the function $\aleph(\alpha),r(n,\epsilon,X)$ and $s(n,\epsilon,X)$ (see [28] p. 173 Theorem 7.7). This property indicates that $D(s,T)$ also can be defined by the spanning sets and separated sets.

Proposition 3.1 *Let T be a continuous transformation on X and $s\ge 0$ a real number. Then*

$$\begin{aligned}D(s,T)&=\lim_{\epsilon\to 0}\limsup_{n\to\infty}\frac{1}{n^s}\log r(n,\epsilon,X)\\&=\lim_{\epsilon\to 0}\limsup_{n\to\infty}\frac{1}{n^s}\log s(n,\epsilon,X).\end{aligned}$$

In [5], the author proved that the s-topological entropy $D(s,T)$ shares the following property.

Proposition 3.2 (i) *The map $s>0\mapsto D(s,T)$ is positive and decreasing with s.*

(ii) *There exists $s_0\in[0,+\infty]$ such that*

$$D(s,T)=\begin{cases}+\infty & \text{if } 0<s<s_0\\ 0 & \text{if } s>s_0.\end{cases}$$

(iii) $D(s,T^m)\le m^sD(s,T)$.

Proposition 3.2(ii) indicates that the value of $D(s,T)$ jumps from infinity to 0 at the two sides of some point s_0, which is similar to a fractal measure. Analogous to the fractal dimension, we can define the entropy dimension of T as the following

$$D(T) = \sup\{s > 0 : D(s,T) = \infty\} = \inf\{s > 0 : D(s,T) = 0\}.$$

3.2 Calculation of $D(s,T)$ and $D(T)$

This section gives some ways to calculate the s-topological entropy of some classes of dynamical systems. Firstly, we obtain the entropy dimensions of the homeomorphisms of the unit circle and the contractive continuous maps.

Theorem 3.1 *Assuming that $T : K \to K$ is a homeomorphism of the unit circle K, then the entropy dimension $D(T) = 0$.*

Proof Without loss of generality, suppose the circle has length 1. First, choose $\epsilon > 0$ small enough such that if $d(x,y) \le \epsilon$ for all $x, y \in K$, then $d(T^{-1}x, T^{-1}y) \le \frac{1}{4}$.

Using the definition of spanning sets for K, it is clear that

$$r(1,\epsilon,K) \le \left[\frac{1}{\epsilon}\right] + 1,$$

where $[\frac{1}{\epsilon}]$ is the integer part of $\frac{1}{\epsilon}$.

We claim that $r(n,\epsilon,K) \le n([\frac{1}{\epsilon}]+1)$. The procedure of the proof of this claim can be seen in Walter's book [28]. Therefore, for any positive value s,

$$\limsup_{n\to\infty} \frac{1}{n^s} \log r(n,\epsilon,K) \le \limsup_{n\to\infty} \frac{\log n([\frac{1}{\epsilon}]+1)}{n^s} = 0,$$

which implies that $D(T) = 0$. □

Theorem 3.2 ([5]) *If $T : X \to X$ is a contractive continuous map, then entropy dimension $D(T) = 0$.*

For example, the identity map and isometries on a compact space are contractive. This implies their entropy dimensions are zero.

In the following, we will give the formulas of s-topological entropies of the piecewise monotone continuous map on the unit interval and the shift map on the subshift space.

Definition 3.3 An interval map $f : [0,1] \to [0,1]$ is called piecewise monotone continuous if there exist points $0 = a_0 < a_1 < \cdots < a_N = 1$ such that $f|_{(a_{i-1},a_i)}$ is continuous and monotone.

Definition 3.4 Let f be a piecewise monotone continuous map. If J is a maximal interval on which $f|_J$ is continuous and monotone, then $f : J \to f(J)$ is called a branch or lap of f. The lapnumber, $l(f)$, is the number of laps of f.

Rothschild [27], Misiurewicz and Szlenk [20] independently obtained the topological entropy formula for a piecewise monotone map (see also [4] and [24]). The following theorem gives a generalized s-topological entropy formula.

Theorem 3.3 *Let $f:[0,1]\to[0,1]$ be a piecewise monotone continuous map and $s>0$ a real number. Then*

$$D(s,f)=\limsup_{n\to\infty}\frac{\log l(f^n)}{n^s}. \tag{3.2}$$

Proof Firstly, we will show

$$D(s,f)\le\limsup_{n\to\infty}\frac{\log l(f^n)}{n^s}.$$

In fact, let $\epsilon>0$ be an arbitrary number. For any branch J of f^n, we can choose a finite set $S_0\subset[0,1]$ such that S_0 contains the two endpoints of $f^n(J)$ and all the neighboring points in $f^n(S_0)$ lie less than ϵ apart, we can choose S_0 such that $\#S_0<2/\epsilon$ since the length of $[0,1]$ is 1. For any $x\in J$, find two neighboring points $y',y''\in S_0$ such that $y'\le x<y''$. Then $|f^n(y')-f^n(x)|<\epsilon$ and $|f^n(y'')-f^n(x)|<\epsilon$. Next, consider whether the distances between $f^{n-1}(x)$ and $f^{n-1}(y')$, $f^{n-1}(y'')$ are less than ϵ. If $|f^n(y')-f^n(x)|>\epsilon$ and $|f^n(y'')-f^n(x)|>\epsilon$, insert a new point $y\in(y',y'')$ such that $|f^{n-1}(y)-f^{n-1}(x)|<\epsilon$. No more than $2/\epsilon$ points (denote by S_1) need to be inserted this way for all points in J. Repeat this process from $f^{n-2}(J)$ to J, we can obtain a finite set $S:=\bigcup_{i=0}^n S_i$ whose cardinality is at most $2(n+1)/\epsilon$. From the construction of S, we know S is an (n,ϵ)-spanning set of J. Thus, $r(n,\epsilon,[0,1])\le 2(n+1)l(f^n)/\epsilon$. Proposition 3.1 shows that

$$\begin{aligned}D(s,f)&\le\lim_{\epsilon\to0}\limsup_{n\to\infty}\frac{\log\frac{2(n+1)l(f^n)}{\epsilon}}{n^s}\\&\le\lim_{\epsilon\to0}\limsup_{n\to\infty}\frac{\log l(f^n)}{n^s}+\lim_{\epsilon\to0}\limsup_{n\to\infty}\frac{\log\frac{2(n+1)}{\epsilon}}{n^s}\\&=\lim_{\epsilon\to0}\limsup_{n\to\infty}\frac{\log l(f^n)}{n^s}.\end{aligned}$$

In the following, we will prove the inverse inequality. For a fixed $m\in\mathbb{N}$, let $g=f^m:[0,1]\mapsto[0,1]$. Since $\{\frac{\log l(g^n)}{(mn)^s}\}$ is a subsequence of $\{\frac{\log l(f^n)}{n^s}\}$,

$$m^s\cdot\limsup_{n\to\infty}\frac{\log l(f^n)}{n^s}\ge\limsup_{k\to\infty}\frac{\log l(g^k)}{k^s}. \tag{3.3}$$

For $l(f^{n+k})\le l(f^n)l(f^k)$, we obtain

$$\begin{aligned}m^s\cdot\limsup_{n\to\infty}\frac{\log l(f^n)}{n^s}&\le\limsup_{n\to\infty}\frac{m^s}{n^s}\log\left(l(g^{[\frac{n}{m}]})l(f^{n-[\frac{n}{m}]m})\right)\\&\le\limsup_{n\to\infty}\frac{m^s}{n^s}\left(\log l(g^{[\frac{n}{m}]})+\max_{0\le i\le m-1}\{\log l(f^i)\}\right)\\&\le\limsup_{k\to\infty}\frac{\log l(g^k)}{k^s}.\end{aligned}$$

Combining (3.3) and the above, it follows

$$m^s\cdot\limsup_{n\to\infty}\frac{\log l(f^n)}{n^s}=\limsup_{k\to\infty}\frac{\log l(g^k)}{k^s}. \tag{3.4}$$

Next, consider the map g. Denote by $\{[a_0, a_1], [a_1, a_2], \dots, [a_{N-1}, a_N]\}$ ($a_0 = 0, a_N = 1$) the collection of the intervals of monotonicity formed by g. Let

$$\alpha = \left\{[a_0, a_1), (a_1, a_2), \dots, (a_{N-1}, a_N], \left(\frac{a_0 + a_1}{2}, \frac{a_1 + a_2}{2}\right), \dots, \right.$$
$$\left. \left(\frac{a_{N-2} + a_{N-1}}{2}, \frac{a_{N-1} + a_N}{2}\right)\right\}.$$

Therefore, α is an open cover of [0, 1] and

$$l(g^k) \le \aleph\left(\bigvee_{i=0}^{k-1} g^{-i}\alpha\right)$$

for any $k \in \mathbb{N}$. Combining (3.4), we have

$$m^s \cdot \limsup_{n\to\infty} \frac{\log l(f^n)}{n^s} \le \limsup_{k\to\infty} \frac{\log \aleph(\bigvee_{i=0}^{k-1} g^{-i}\alpha)}{k^s} \le D(s, g).$$

Combining Proposition 3.2(iii), we obtain

$$\limsup_{n\to\infty} \frac{\log l(f^n)}{n^s} \le D(s, f).$$

Therefore, (3.2) holds. □

From the formula (3.2), we have the following corollary.

Corollary 2 *Let $f : [0, 1] \to [0, 1]$ be a piecewise monotone continuous map and $s > 0$ a real number. For any $m \in \mathbb{N}$, we have*

$$D(s, f^m) = m^s D(s, f) \tag{3.5}$$

and

$$D(f^m) = D(f).$$

Proof Since

$$D(s, f^m) = m^s \limsup_{n\to\infty} \frac{\log l(f^{mn})}{m^s n^s} \le m^s \limsup_{n\to\infty} \frac{\log l(f^n)}{n^s} = m^s D(s, f),$$

then $D(s, f^m) \le m^s D(s, f)$.

On the other hand, for any $n \in \mathbb{N}$, we can write $n = km + i$ $(0 \le i < m)$. Then

$$l(f^n) = l(f^{km+i}) \le l(f^{km}) l(f^i).$$

Thus

$$D(s, f) = \limsup_{n\to\infty} \frac{\log l(f^n)}{n^s} \le \limsup_{k\to\infty} \frac{\log l(f^{km})}{(km)^s} = \frac{1}{m^s} D(s, f^m).$$

That is, $m^s D(s, f) \le D(s, f^m)$. Therefore, (3.5) holds, which implies $D(f^m) = D(f)$. □

Let us consider the symbolic dynamics. For a fixed $m \in \mathbb{N}$, we have the full shift space $\Sigma_m = \{0, 1, \dots, m-1\}^{\mathbb{N}}$, the left shift map $\sigma : \Sigma_m \to \Sigma_m$, and the following metric on Σ_m:

$$d(x, y) = \sum_{i \geq 1} 2^{-i} \delta(x_i, y_i),$$

where $\delta(a, b) = 1$ if $a \neq b$ and $\delta(a, b) = 0$ otherwise; $x = (x_i)_{i \geq 1}, y = (y_i)_{i \geq 1} \in \Sigma_m$. The symbolic space (Σ_m, d) is a compact metric space and σ is continuous. A set $S \subset \Sigma_m$ is called a shift space provided it is closed and is invariant under the shift map σ, we refer to such S's as subshifts.

Let $S \subset \Sigma_m$ be a subshift space. Let $c_1 c_2 \cdots c_n \in \{0, 1, \dots, m-1\}^n$, the set

$$C(c_1 c_2 \cdots c_n) := \{s \in S : s_1 s_2 \cdots s_n = c_1 c_2 \cdots c_n\}$$

is called a cylinder set of length n. Denote by $\mathcal{C}_n(S)$ the collection of nonempty n-cylinders (i.e., cylinder sets of length n). Denote by

$$p(n) := \#\mathcal{C}_n(S)$$

the number of n-cylinders in S, for $n \in \mathbb{N}$.

Theorem 3.4 *Let $S \subset \Sigma_m$ be a subshift space. Then*

$$D(s, \sigma) = \limsup_{n \to \infty} \frac{\log p(n)}{n^s}.$$

Proof We easily know

$$p(n) \leq p(n + l) \leq p(n) p(l), \tag{3.6}$$

where $n, l \in \mathbb{N}$.

For any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $2^{-N-1} < \epsilon \leq 2^{-N}$. Let C_{n+N+2} be any $n + N + 2$ cylinder, then $\sigma^n(C_{n+N+2})$ is an $N + 2$ cylinder. Therefore, if $s, t \in C_{n+N+2}$, then $d_n(s, t) \leq 2^{-N-1} < \epsilon$. Hence, we obtain a sequence by choosing a point from each $n + N + 2$ cylinder; this sequence is an (n, ϵ)-spanning set. Thus, $r(n, \epsilon, S) \leq p(n + N + 2)$. Propositions 3.1 and (3.6) show that

$$\begin{aligned} D(s, \sigma) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n^s} \log r(n, \epsilon, S) &\leq \lim_{N = N(\epsilon) \to \infty} \limsup_{n \to \infty} \frac{1}{n^s} \log p(n + N + 2) \\ &\leq \lim_{N \to \infty} \limsup_{n \to \infty} \frac{1}{n^s} (\log p(n) + \log p(N + 2)) \\ &= \limsup_{n \to \infty} \frac{\log p(n)}{n^s}. \end{aligned}$$

On the other hand, if $s \in C_{n+N}$ and $t \in \hat{C}_{n+N} \neq C_{n+N}$, then there exists $0 \leq i \leq n$ such that $d(\sigma^i(s), \sigma^i(t)) \geq 2^{-N+1} > \epsilon$, so $d_n(s, t) > \epsilon$. Therefore, any (n, ϵ)-spanning set contains at least one point of each $n + N$ cylinder. Proposition 3.1 and (3.6) show that

$$D(s,\sigma)=\lim_{\epsilon\to 0}\limsup_{n\to\infty}\frac{1}{n^s}\log r(n,\epsilon,S)\geq \lim_{N=N(\epsilon)\to\infty}\limsup_{n\to\infty}\frac{\log p(n+N+2)}{n^s}$$
$$\geq \lim_{N\to\infty}\limsup_{n\to\infty}\frac{\log p(n)}{n^s}$$
$$=\limsup_{n\to\infty}\frac{\log p(n)}{n^s}.$$

Therefore,

$$D(s,\sigma)=\limsup_{n\to\infty}\frac{\log p(n)}{n^s}. \qquad \square$$

3.3 Examples

1. Tent map

Let $f_\lambda := \lambda \min\{x, 1-x\}$ be a family of functions defined on [0, 1], where $0<\lambda\leq 2$. These functions are so-called tent maps. The topological entropy is equal to $\log\lambda$ when $1<\lambda\leq 2$ and 0 when $0<\lambda\leq 1$.

When $1<\lambda\leq 2$, by the definition of entropy dimension, we know $D(f_\lambda)=1$. When $0<\lambda\leq 1$, f_λ is contractive and Theorem 3.2 shows that $D(f_\lambda)=0$.

2. Subshift with any given entropy dimension

Let $0<\alpha<1$. In [6], the author constructed an infinite word $w\in\Sigma_2=\{0,1\}^{\mathbb{N}}$ whose complexity function $L_w(n)$ satisfies the following

$$L_w(n)\sim 2^{n^\alpha}, \tag{3.7}$$

where $L_w(n)$ is the number of finite words with length n that occur as a block of consecutive letters in w. Let $\mathcal{G}_n=\{C(x_1x_2\cdots x_n): x_1x_2\cdots x_n \text{ occurs in } w\}\subset\Sigma_2$ and $X_n=\bigcup_{I\in\mathcal{G}_n} I$. The set X_n is the union of at most 2^n n-cylinder sets. Denote

$$X=\bigcap_{n=1}^{\infty}X_n.$$

Theorem 3.5 *The space (X,σ) is a subshift of Σ_2 and*

$$D(\sigma)=\alpha.$$

Proof On one hand, since the set X_n is the finite union of n-cylinders and the cylinders are closed sets, then X_n is closed. Thus, X is closed.

On the other hand, we claim that the shift σ is closed in X, that is, $x=x_1x_2\cdots x_n\cdots\in X$ implies $\sigma x\in X$. In fact, for any $n\in\mathbb{N}$, $C(x_1x_2\cdots x_n)\in\mathcal{G}_n$, then $x_1x_2\cdots x_n$ occurs in w. This implies $x_2\cdots x_n$ occurs in w, then $C(x_2\cdots x_n)\in\mathcal{G}_{n-1}$. Then we have $\sigma x\in X$.

Therefore, X is a subshift of Σ_2. The construction of X and (3.7) shows the number of n-cylinders of X $p(n)\sim 2^{n^\alpha}$. We obtain that $D(\alpha,\sigma)=1$ by Theorem 3.4. Therefore, $D(\sigma)=\alpha$ according to the definition of entropy dimension. $\square$

3. An example with zero entropy and full entropy dimension

Next, construct a dynamical system (X,T) with zero topological entropy but full entropy dimension, i.e., $h(T)=0$ and $D(T)=1$. This answers a question raised by Carvalho in [5] where he asked whether $D(T)=1$ implies $h(T)>0$.

Let $\{a_n\}$ be an increasing sequence of real numbers with $a_n \to 1$ as $n \to \infty$. According to Cassaigne's [6] method, we can construct a sequence of words $\{w_n\}$ such that the complexity functions $L_{w_n}(m) \sim 2^{m^{a_n}}$. Let $\mathcal{G}_m = \{C(x_1x_2\cdots x_m): \text{ there exists } i \in \mathbb{N} \text{ such that } x_1x_2\cdots x_m \text{ occurs in } w_i\} \subset \Sigma_2$ and $X_m = \bigcup_{I\in\mathcal{G}_m} I$. Denote

$$X = \bigcap_{m=1}^{\infty} X_m.$$

Theorem 3.6 *The space* (X, σ) *is a subshift of* Σ_2 *and*

$$h(\sigma) = 0, \qquad D(\sigma) = 1.$$

Proof Since X_m is the union of at most 2^m m-cylinders, then X is a closed set. As in the second example, the shift σ is closed in X, so X is a subshift of Σ_2.

Since the number of m-cylinders of X $p(m) \geq 2^{m^{a_n}}$ for any $n \geq 1$, then $D(\sigma) \geq a_n$ by Theorem 3.4 and the definition of the entropy dimension. Letting $n \to \infty$, we have $D(\sigma) \geq \limsup_{n\to\infty} a_n = 1$. Therefore, $D(\sigma) = 1$.

It is left to prove that $h(\sigma) = 0$. Let

$$\mathcal{C}_{n,m} = \{C(x_1, \ldots, x_m) : x_1 \ldots x_m \text{ occurs in } w_n\}$$

and $Y_{n,m} = \bigcup_{I\in\mathcal{C}_{n,m}} I$. Denote

$$Y_n = \bigcap_{m=1}^{\infty} Y_{n,m}.$$

The second example shows that $D(\sigma|_{Y_n}) = a_n$, which implies $h(\sigma|_{Y_n}) = 0$.

Therefore,

$$h(\sigma|_{\cup_{n=1}^{\infty} Y_n}) = 0.$$

Note that $X \subset \bigcup_{n=1}^{\infty} Y_n$, so $h(\sigma|_X) = 0$. □

3.4 Properties of $D(s, T)$

Definition 3.5 Let $T_1 : X_1 \to X_1$ and $T_2 : X_2 \to X_2$ be two continuous maps. A continuous map $\phi : X_1 \to X_2$ is called a topological semi-conjugacy from T_1 to T_2, if $\phi \circ T_1 = T_2 \circ \phi$ and $\phi(X_1) = X_2$. If $\phi : X_1 \to X_2$ is a homeomorphism with $\phi \circ T_1 = T_2 \circ \phi$, we call ϕ a topological conjugacy.

Theorem 3.7 *Let* X *and* Y *be compact metric spaces with metrics* d *and* d', *respectively. Let* $T_1 : X \to X$ *and* $T_2 : Y \to Y$ *be semi-conjugate by* $\phi : X \to Y$. *Then* $D(s, T_1) \geq D(s, T_2)$ *for any* $s > 0$.

Proof Since X is compact and ϕ is continuous, ϕ is uniform continuous. So for any given $\epsilon > 0$, there exists $\delta > 0$ such that $d(x_1, x_2) \geq \delta$ whenever $d'(\phi(x_1), \phi(x_2)) \geq \epsilon$. Let $E(n, \epsilon, Y) \subset Y$ be a maximal (n, ϵ)-separated set for T_2, i.e., $\#E(n, \epsilon, Y) = s(n, \epsilon, Y)$. Form the set $E(n, \delta, X) \subset X$ by taking one $x \in \phi^{-1}(y)$ for each $y \in E(n, \epsilon, Y)$, then $\#E(n, \delta, X) = \#E(n, \epsilon, Y)$.

We claim that $E(n,\delta,X)$ is a (n,δ)-separated set for T_1. In fact, for any $x_1, x_2 \in E(n,\delta,X)$, $\phi(x_1), \phi(x_2) \in E(n,\epsilon,Y)$. Thus,

$$d'_n(\phi(x_1),\phi(x_2)) = \max_{0\le i<n} d'(T_2^i\phi(x_1), T_2^i\phi(x_2)) \ge \epsilon,$$

that is, there exists $0 \le i_0 < n$ such that $d'(T_2^{i_0}\phi(x_1), T_2^{i_0}\phi(x_2)) \ge \epsilon$. Since $\phi \circ T_1^{i_0} = T_2^{i_0} \circ \phi$, then $d(T_1^{i_0}x_1, T_1^{i_0}x_2) \ge \delta$ by the uniform continuity of ϕ. That is, $d_n(x_1,x_2) \ge \delta$.

Therefore, $s(n,\delta,X) \ge \#E(n,\delta,X) = \#E(n,\epsilon,Y) = s(n,\epsilon,Y)$. This shows that $D(s,T_1) \ge D(s,T_2)$ for any $s > 0$ by Proposition 3.1. □

If $\phi : X \to Y$ is a conjugacy, we apply Theorem 3.7 to $\phi : X \to Y$ and $\phi^{-1} : Y \to X$ to obtain the following corollary. This indicates that the s-topological entropy is still an invariant of topological conjugacy for any $s > 0$.

Corollary 3 *Let (X,d) and (Y,d') be compact metric spaces. $T_1 : X \to X$ and $T_2 : Y \to Y$ are two continuous maps. If (X,d,T_1) is conjugate to (Y,d',T_2), then $D(s,T_1) = D(s,T_2)$ for any $s > 0$.*

Theorem 3.8 *Let (X_1,d_1) and (X_2,d_2) be compact metric spaces. $T_1 : X_1 \to X_1$ and $T_2 : X_2 \to X_2$ are two continuous maps. Define a metricd on $X_1 \times X_2$ by $d((x_1,x_2),(y_1,y_2)) = \max\{d_1(x_1,y_1), d_2(x_2,y_2)\}$ a transformation by $(T_1 \times T_2)(x,y) = (T_1x, T_2y)$. Then $D(s,T_1 \times T_2) = D(s,T_1) + D(s,T_2)$ for any $s > 0$.*

Proof Since balls in the product metric d are products of balls on X_1 and X_2, the same is true for balls in the n-Bowen metric d_n. Hence,

$$r(n,\epsilon,X_1 \times X_2) \le r(n,\epsilon,X_1) \cdot r(n,\epsilon,X_2).$$

Thus, $D(s,T_1 \times T_2) \le D(s,T_1) + D(s,T_2)$.

On the other hand, the product of any (n,ϵ)-separated set in X_1 for T_1 and any (n,ϵ)-separated set for T_2 is an (n,ϵ)-separated set for $T_1 \times T_2$. Thus,

$$s(n,\epsilon,X_1 \times X_2) \ge s(n,\epsilon,X_1) \cdot s(n,\epsilon,X_2),$$

which implies that $D(s,T_1 \times T_2) \ge D(s,T_1) + D(s,T_2)$. □

4 Remark

For a continuous function T of a compact metric space (X,d), the well-known Variational Principle of entropy shows the relationship between topological entropy and measure-theoretical entropy as follows:

$$h(T) = \sup_{\mu \in M(X,T)} h_\mu(T)$$

where $M(X,T)$ is the collection of invariant measures of X. Dou, Huang, and Park [8] considered a class of symbolic dynamical systems X_α's associated with the special kind of infinite sequence u_α. They demonstrated that those topological dynamical systems are uniquely ergodic and the measure-theoretic entropy dimensions are identically zero. However, it was shown that the topological entropy dimension of the symbolic dynamical system X_α is α where $0 < \alpha < 1$. This implies that the variational principle does not hold for the entropy dimension even when the T-invariant partition $\mathcal{A}$ is equal to the whole space X.

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